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Clifford Modules and Generalized Dirac Operators

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We present a review of generalized Dirac operators with special focus on the Bochner–Lichnerowicz–Weitzenböck (BLW) decomposition of the square of a Dirac operator. We discuss how a specific Dirac operator, which we call the Pauli–Dirac operator, is related to electric charge in a similar way that the Dirac–Yukawa operator is related to mass. By joining together both operators, we end up with a new Dirac operator which we call the Pauli–Dirac–Yukawa (PDY) operator. This operator, which plays a crucial role in the geometrical description of the Standard Model of particle physics, has a specific structure. We discuss the BLW decomposition of the "real form" (i.e., the real part of the modulus) of the PDY.

1. INTRODUCTION

The basic property of a Dirac operator is that it is the square root of a Laplacian-type operator. Dirac's idea was to introduce a first-order differential operator that can be considered as the square root of a second-order operator similar to the complex number $iu - v$, which is the square root of $u^2 + v^2 = (iu - v)(-iu - v)$.

We will start with a summary of the main features of generalized Dirac operators and review the Bochner–Lichnerowicz–Weitzenböck (BLW) decomposition of the square of a certain class of Dirac operators. This decomposition may be generalized to arbitrary Dirac operators. While there are several decomposition formulas for generalized Dirac operators, we will focus on a specific decomposition formula which is particularly geometric.

In the second part of the paper we discuss two specific generalized Dirac operators, the Dirac–Yukawa operator and the Pauli–Dirac operator. These two operators are of significance because the Dirac–Yukawa operator

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gives rise to the dynamics of the fermions and the Pauli–Dirac operator encodes the dynamics of the gauge fields. It is then natural to bring both Dirac operators together to define a new Dirac operator, called the Pauli–Dirac–Yukawa operator (PDY). In fact, this operator not only defines the fermionic action, but also the bosonic functional of the Standard Model of elementary particles (Tolksdorf, 1998). For this purpose, one needs the BLW decomposition of the "real form" of the Pauli–Dirac–Yukawa operator. This decomposition has a generic form which we will discuss, and which finally will represent a synthesis of both parts of the present paper. In the light of Dirac operators, the two different concepts of mass and of electric charge actually correspond to two different $\mathbb{Z}_2$-gradings. As we will see in the second part of the paper, this superstructure is very basic for the PDY to be a Dirac operator in the generalized sense according to Atiyah and Singer (1963) and Atiyah and Bott (1968).

2. A BRIEF HISTORICAL SURVEY

When Dirac introduced his equation in 1928/30, he aimed at relativistically generalizing the Schrödinger wave equation of quantum mechanics (Dirac, 1928, 1930). With respect to this original goal, he actually failed. However, his equation correctly describes some fundamental properties of an electron, e.g., its half-integer angular momentum and its magnetic moment. Moreover, his equation predicted the existence of a new particle, the positron, of exactly the same mass as the electron, but with opposite electric charge. Today, the existence of "antiparticles" is well established. However, at the time when Dirac introduced his equation, only a few elementary particles were known and the prediction of a new particle was extremely speculative. When the positron was actually found in 1932/33 by Anderson (Anderson, 1932, 1933) this was probably one of greatest triumphs in theoretical physics. Dirac's equation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad (i\gamma^\mu\partial_\mu - m)\psi(x) = 0, \qquad \not{\partial} \equiv \gamma^\mu\partial_\mu \tag{1}$$

became one of the most fundamental equations in physics. The fundamental relations that the Dirac matrices γ^μ ($\mu = 0, \ldots, 3$) have to satisfy are known as the Clifford algebra $\mathcal{C}_{1,3}$ over the indefinite quadratic space $\mathbb{R}^{1,3}$. One of the most basic properties of the Dirac operator is that it can be considered as the "square root" of the wave (Laplace) operator $g^{\mu\nu}\,\partial_\mu\partial_\nu$. When using the physical "principle of minimal coupling," it becomes more or less straightforward to generalize $\not{\partial}$ to a "gauge-covariant" operator, which is known today as the "twisted Dirac operator" $\not{\partial}_A$. Generalizing the Dirac operator to nonflat spacetimes, in contrast, was not evident before Cartan introduced his "repere mobile" formalism. The mathematical question that arose was how to convert the Dirac operator to a globally defined first-order differential operator on a general (pseudo) Riemanian manifold. For this purpose, the notion of a spin manifold was introduced and a new door was opened to study geometrical and topological properties of manifolds using a local differential operator. Today this is called spin geometry. The Laplace operator encodes geometrical and topological information (according to Hodge's and de Rham's theory and the heat kernel associated with the Laplace operator). However, the Dirac operator proved to be a more fundamental object in this respect. Since the Laplacian is a symmetric operator, its index vanishes. Moreover, based on the fact that every Fredholm operator on a compact manifold can in some sense be deformed into a Dirac operator, the latter incorporates a very deep relation between analysis and topology. The Dirac operator joins seemingly different mathematical fields like geometry, topology, and analysis. This is demonstrated very impressively by the index theorems by Atiyah and Singer and the corresponding generalizations (see, for instance, Berline and Getzler and Verne, 1996). Physicists used the Dirac equation to develop quantum field theory. This theory permits describing both the photon (gauge field) and the electron/positron (spinor) by use of the same formalism. This formalism fundamentally differs from quantum mechanics in that the number of particles within a certain physical process is no longer a conserved quantity. This is an experimentally well-established fact which has always been encoded in Dirac's equation and in the notion of particles and antiparticles. Another interesting aspect in quantum field theory was that physicists found certain relations, called anomalies, which turned out to be the index of a twisted Dirac operator. The operator invented by Dirac in 1928 has not only joined different fields in mathematics and physics, but has also brought more closely together physics and mathematics as a whole.

So far we have only considered twisted Dirac operators $\not{D}_A$ defined on a spin manifold M as a mathematical generalization of the operator originally introduced by Dirac. In fact, this type of operator plays a fundamental role in spin geometry. The point is that every representation space $\mathscr{E}$ (bundle) of the Clifford algebra (bundle) over a spin manifold is isomorphic to a twisted spinor bundle over M (details of the following will be given in the next section). From a physical point of view, a section into such a twisted spinor bundle geometrically models a "particle with internal degrees of freedom" (a particle with generalized charges). However, a twisted spinor bundle is but a special case (but may be the most important one!) of a general Clifford module bundle $\mathscr{E}$ over an arbitrary manifold. There are no topological obstructions for a Clifford algebra bundle to exist. Indeed, the Clifford bundle $Cl(M)$ may be the most natural nontrivial algebra bundle associated with a given (pseudo) Riemannian manifold (M, g). If the dimension of M is even, it turns out that there exists a distinguished class of connections called Clifford connections on every Clifford module bundle $\mathscr{E}$ and a distinguished class of Dirac operators (also denoted by $\not{\partial}_A$) uniquely associated with these connections. Atiyah (1988) and Lichnerowicz (1963) may have been the first to introduce the notion of generalized Dirac operators D on an arbitrary Clifford module bundle $\mathscr{E}$. Such a generalized Dirac operator can always be written as

$$\mathrm{D} = \not{\partial}_A + \Phi \tag{2}$$

where Φ is a zero-order operator. Interestingly, operators of this form are very well known in physics for special Φ's. In fact, one such operator has been introduced by physicists to gauge-invariantly describe the masses of the fermions (Yukawa coupling). This operator Φ generalizes the positive real number m in Dirac's equation (1). The Dirac operator for these particular Φ's is called the Dirac–Yukawa operator. In turn, the Dirac–Yukawa operator plays a fundamental role in Connes' noncommutative geometry (Connes, 1994). Another interesting operator of the form of a generalized Dirac operator that was introduced by physicists is given by $\Phi := i\gamma^{\mu}\gamma^{\nu}\,\mathrm{F}_{\mu\nu}$. Here, F denotes the curvature of the connection form A. In this case we call the generalized Dirac operator the "Pauli–Dirac operator," an operator originally introduced to correctly describe the anomalous magnetic moment of protons which significantly differs from the value 2 as can be deduced from $\not{\partial}_A$. It can be shown that the Wodzicki residue of the pseudo-differential operator D^{-2} (in more physical terms: the "trace of the propagator" of the Pauli–Dirac operator) is proportional to the combined Einstein–Hilbert–Yang–Mills functional (Ackermann and Tolksdorf, 1996a, b). We will consider the Dirac–Yukawa and the Pauli–Dirac operators in more detail in Section 5.

Generalized Dirac operators are not only interesting objects used in mathematics and physics, they also serve as a joining object to bring mathematics and physics more closely together.

3. THE CLIFFORD FRAME

For this section we refer to the book by Berlin *et al.* (1996).

Let us denote by $M^{(p,\,q)}$ a smooth manifold of dimension $m := p + q$ together with a smooth, nondegenerate, symmetric, bilinear form g of signature $s := p - q$. A structure naturally associated with $M^{(p,q)}$ is the *Clifford bundle*: $Cl(M) \xrightarrow{\tau} M$. Thus, the Clifford bundle is an algebra bundle with standard fiber equal to the Clifford algebra $\mathscr{C}_{p,q}$. Also let us denote by $\mathscr{E} \xrightarrow{\pi} M$ a smooth ($\mathbb{Z}_2$-graded) vector bundle and by $\Gamma(\mathscr{E})$ the $C^{\infty}(M)$ module of smooth sections into $\mathscr{E}$. A *generalized Laplacian H*: $\Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ is a second-order differential operator such that for every smooth function $f \in C^{\infty}(M)$, we have

$$[[H, f], f] = \pm 2g(df, df) \tag{3}$$

A *generalized Dirac operator* (operator of Dirac type) is any first-order differential operator $\mathrm{D}: \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ such that D^2 is a generalized Laplacian. In the case where the vector bundle $\mathscr{E}$ is $\mathbb{Z}_2$-graded, we demand the operator D to be odd. It can be shown that an (odd) first-order differential operator D is a generalized Dirac operator if and only if it satisfies the fundamental relation

$$[\mathrm{D}, f] = \gamma(df) \tag{4}$$

such that $(\gamma(df))^2 = \pm g(df, df)$ holds true for every smooth function $f \in C^\infty(M)$ (Berline *et al.*, 1996). According to the relation (4), such an operator D induces a Clifford map $\gamma: T^*M \to \mathrm{End}(\mathscr{E})$ and thus $(\mathscr{E}, \gamma)$ becomes a *Clifford module* bundle over (M, g).

In what follows, we will focus on the even-dimensional case $(M, g) \equiv M^{(0,2n)}$, where $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ denotes a $\mathbb{Z}_2$-graded Clifford module bundle over M with respect to the Clifford mapping γ. We assume the induced Clifford action (which is also denoted by γ) to be even, i.e., $\gamma(Cl^+)\mathscr{E}^\pm \subset \mathscr{E}^\pm$ and $\gamma(Cl^-)\mathscr{E}^\pm \subset \mathscr{E}^\mp$. The Clifford algebra is simple and the algebra bundle of bundle endomorphisms on $\mathscr{E}$ decomposes as follows:

$$\mathrm{End}(\mathscr{E}) \simeq Cl(M) \otimes \mathrm{End}_{Cl}(\mathscr{E}) \tag{5}$$

where $\mathrm{End}_{Cl}(\mathscr{E}) := \{\sigma \in \mathrm{End}(\mathscr{E}) | [\gamma(a), \sigma]_\pm = 0, \text{ for all } a \in Cl(M)\}$ denotes the "supercommutant" with respect to the Clifford map γ. Because of this decomposition, a distinguished class of connections exists on any Clifford module bundle $(\mathscr{E}, \gamma)$ which is characterized by the condition

$$[\nabla^{\mathscr{E}}_\xi, \gamma(a)] = \gamma(\nabla^{Cl}_\xi a) \tag{6}$$

for all smooth vector fields $\xi \in \Gamma(TM)$ and smooth sections $a \in \Gamma(Cl(M))$. Here, ∇^{Cl} denotes the induced Levi-Civita connection on the Clifford bundle. In other words, connections satisfying relation (6) are distinguished by their compatibility with the Clifford action γ. They are thus called *Clifford connections*. If (M, g) denotes a *spin manifold*, any Clifford module bundle $(\mathscr{E}, \gamma)$ is (isomorphic to) a *twisted spinor bundle* over (M, g). That is, there exists a vector bundle $\mathrm{E} \xrightarrow{\pi_\mathrm{E}} M$ such that $\mathscr{E} \simeq \mathrm{S} \otimes \mathrm{E}$, where S denotes the (total space of the) spinor bundle over M once a spin structure has been fixed. As a consequence, the (complexification of the) Clifford bundle will be identified with End(S) and the supercommutant $\mathrm{End}_{Cl}(\mathscr{E})$ with End(E). Therefore, in the case of a spin manifold, we obtain $\mathrm{End}(\mathscr{E}) \simeq \mathrm{End(S)} \otimes \mathrm{End(E)}$. It is easily verified that any Clifford connection is a tensor product connection, i.e., it reads $\nabla^{\mathscr{E}} = \nabla^\mathrm{S} \otimes \mathrm{Id}_\mathrm{E} + \mathrm{Id}_\mathrm{S} \otimes \nabla^\mathrm{E}$, where ∇^S denotes the spin connection on the spinor bundle S and ∇^E denotes any connection on the corresponding vector bundle E. Therefore, the Clifford connections on a twisted spinor bundle are parametrized by the connections on the vector bundle E. This holds true even in the more general case of an arbitrary Clifford module. To make this more precise, let us denote by $\mathcal{A}(\mathcal{E})$ the set of all (even) connections on $\mathcal{E}$. This set is an affine space which is modeled on the vector space of even one-forms $\Omega^1(M, \mathrm{End}^+(\mathcal{E}))$. Because of the decomposition (5), there is a certain affine subspace $\mathcal{A}_{Cl}(\mathcal{E}) \subset \mathcal{A}(\mathcal{E})$ of the set of all connections on $\mathcal{E}$. This affine subspace is correspondingly modeled on the vector space $\Omega^1(M, \mathrm{End}^+_{Cl}(\mathcal{E}))$. As a result, the Clifford connections on an arbitrary Clifford module bundle are parametrized by the vector space $\Omega^1(M, \mathrm{End}^+_{Cl}(\mathcal{E}))$. Let $\nabla^{\mathcal{E}}$ be a connection on the Clifford module bundle $(\mathcal{E}, \gamma)$. A well-known construction to associate a Dirac operator with a given connection is

$$\not\nabla^{\mathcal{E}}: \quad \Gamma(\mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} \Gamma(T^*M \otimes \mathcal{E}) \hookrightarrow \Gamma(Cl(M) \otimes \mathcal{E} \xrightarrow{\gamma} \Gamma(\mathcal{E}) \tag{7}$$

It is straightforward that the operator $\not\nabla^{\mathcal{E}} \equiv \gamma(\nabla^{\mathcal{E}})$ is odd and indeed satisfies the fundamental relation (4). There exists a distinguished class of Dirac operators on a Clifford module bundle, namely Dirac operators associated with Clifford connections $\partial_A \in \mathcal{A}_{Cl}(\mathcal{E})$. In what follows, we will denote such a Dirac operator by $\not\partial_A$, where $A \in \Omega^1 (M, \mathrm{End}^+_{Cl}(\mathcal{E}))$. It is called the *twisted Dirac operator.* In the particular case that (M, g) denotes a spin manifold, the Dirac operators associated with Clifford connections are called *twisted spin Dirac operators*.

We will show how a given Dirac operator can be represented by a connection. Let $(\mathcal{E}, \mathrm{D})$ denote a Clifford module bundle over (M, g), where the Clifford action γ is defined by some Dirac operator D with the help of the fundamental relation (4). Any other Dirac operator is considered to be compatible with the now-defined Clifford action γ if it also satisfies the fundamental relation (4). Let us denote by $\mathcal{D}(\mathcal{E})$ the set of all compatible Dirac operators on the Clifford module $(\mathcal{E}, \mathrm{D})$. This set is again an affine space modeled on the vector space of odd zero-forms $\Omega^0(M, \mathrm{End}^-(\mathcal{E}))$. There is a *canonical one-form* $\xi \in \Omega^1(M, \mathrm{End}^-(\mathcal{E}))$ which satisfies the relations $\nabla^{T^*M\otimes\mathrm{End}(\mathcal{E})}\xi \equiv 0$ for all Clifford connections ∂_A, and $\gamma(\xi) = \mathrm{Id}_{\mathcal{E}}$ (Tolksdorf, 1998). Therefore, we obtain a mapping

$$\begin{aligned} \delta_\xi: \quad \Omega^0(M, \mathrm{End}^-(\mathcal{E})) &\to \Omega^1(M, \mathrm{End}^+(\mathcal{E})) \\ \Phi &\mapsto \xi \wedge \Phi \end{aligned} \tag{8}$$

and thus can associate with every Dirac operator $\tilde{\mathrm{D}} \in \mathcal{D}(\mathcal{E})$ a connection $\tilde{\nabla}^{\mathcal{E}} := \partial_A + \delta_\xi(\tilde{\mathrm{D}} - \not\partial_A)$ such that $\tilde{\mathrm{D}} = \tilde{\not\nabla}^{\mathcal{E}}$. However, since δ_ξ is a right inverse of γ, the correspondence between Dirac operators and connections is not one-to-one. In fact, it can be shown that $\mathcal{D}(\mathcal{E}) \simeq \mathcal{A}(\mathcal{E})/\mathrm{Ker}(\gamma)$ (Tolksdorf, 1998). Consequently, every Dirac operator is associated with a whole class of connections on $\mathscr{E}$. Note that every connection class possesses at most one Clifford connection. Thus, the corresponding connection class of a twisted Dirac operator has a canonical representative. Two connections defining the same Dirac operator are not in general related to each other by a gauge transformation. Although the correspondence between Dirac operators and connections is not one-to-one, there exists a "natural connection" that is associated with a given Dirac operator. Every Dirac operator $D \in \mathscr{D}(\mathscr{E})$ is of the form

$$D = \not{\partial}_A + \Phi \tag{9}$$

with $\Phi \in \Omega^0(M, \mathrm{End}^-(\mathscr{E}))$. In addition, any Dirac operator is associated with a one-form ω_Φ defined by

$$\omega_\Phi := \delta_\xi(\Phi) \tag{10}$$

which we call the *Dirac form*. Let $\partial_A \in \mathscr{A}_{Cl}(\mathscr{E})$ denote the Clifford connection uniquely associated with the twisted Dirac operator $\not{\partial}_A$. We then define the connection

$$\tilde{\nabla}^{\mathscr{E}} := \partial_A + \omega_\Phi \tag{11}$$

and call it the "*natural connection*" associated with the Dirac operator D.

4. THE BLW DECOMPOSITON

We will discuss the BLW decomposition of a generalized Dirac operator. We start by summarizing the decomposition formulas of various specific examples of Dirac operators. The corresponding decomposition formulas are named after Weitzenböck, Bochner, and Lichnerowicz.

In this section, we consider various examples of Dirac operators. By its very definition, the square of a Dirac operator is a generalized Laplacian. It is therefore natural to ask whether these Laplacians have a common structure. In the case of Dirac operators associated with Clifford connections, this will be answered affirmatively through the decomposition formulas named after Bochner, Lichnerowicz, and Weitzenböck. These formulas are special cases of a general decomposition formula which can be applied to arbitrary Dirac operators. We will call this decomposition formula the (generalized) *Bochner–Lichnerowicz–Weitzenböck decomposition formula* (BLW decomposition). While we are aware that various kinds of BLW decomposition formulas for arbitrary operators of the Dirac type exist, we will only discuss a specific decomposition formula of the BLW type in this section. This formula is particularly geometric and it very transparently generalizes the original Lichnerowicz formula.

As a first example, we consider the case of the exterior algebra bundle $\Lambda^*(T^*M) \to M$. The set of sections is denoted by $\Omega^*(M)$, where $\Lambda^*(T^*M) := \bigoplus_{p \in \mathbb{Z}} \Lambda^p(T^*M)$. If we divide $\Lambda^*(T^*M)$ into even and odd forms, it becomes a $\mathbb{Z}_2$-graded vector bundle. Moreover, using, respectively, the exterior and interior multiplication ext_v and int_v, with $v \in TM \simeq_g T^*M \hookrightarrow \Lambda^*(T^*M)$, the $\mathbb{Z}_2$-graded vector bundle $\Lambda^*(T^*M)$ becomes a Clifford module bundle over (M, g). We will use the *Gauss–Bonnet* operator $D := d + d^*$ as Dirac operator, where d^* denotes the formal adjoint of d with respect to the natural scalar product on $\Omega^*(M) \ni \alpha, \beta \mapsto \int_M \alpha \wedge * \beta$. Of course, in this case, the square of D gives the well-known Laplace–Beltrami operator $\Delta := (dd^* + d^*d)$. The Weitzenböck decomposition formula relates this Laplacian to the Laplacian $\nabla^*\nabla$. Here, the latter is a special case of the so-called Bochner Laplacian, for which we will give a more general definition below. The operator ∇ denotes the lifted Levi-Civita connection on $\Lambda^*(T^*M)$, and ∇^* denotes its formal adjoint. It can be shown that the difference of both Laplacians is given by a zero-order differential operator. The *Weitzenböck formula* gives the relation (Rosenberg, 1997; Lawson and Michelson, 1997; Berline *et al.*, 1996)

$$\Delta = \nabla^*\nabla + \mathcal{R} \tag{12}$$

where $\mathcal{R}$ denotes the curvature operator. With respect to an orthonormal frame (e^i) in $T^* M$, it is given by

$$\mathcal{R} = \frac{r_M}{4} + \frac{1}{8} R_{ijkl}\gamma^i\gamma^j\overline{\gamma}^k\overline{\gamma}^l \tag{13}$$

Here, $\gamma^i := \mathrm{ext}_i - \mathrm{int}_i$, $\overline{\gamma}^i := \mathrm{ext}_i, + \mathrm{int}_i$, and, respectively, R_{ijkl} denote the components of the curvature tensor with respect to the orthonormal basis (e^i) and r_M the scalar curvature. Note that $[\gamma^i, \gamma^j]_+ = -2\delta^{ij}$, whereas $[\overline{\gamma}^i, \overline{\gamma}^j]_+ = +2\delta^{ij}$, where we have used the common notation $\gamma^i \equiv \gamma(e)^i)$. If restricted to one-forms, the formula (12) is called the *Bochner formula*. In this case, the curvature operator $\mathcal{R}$ is given by the Ricci curvature and it becomes straightforward to prove that a (compact and orientable) manifold with non-vanishing first de Rham cohomolgy group admits no metric with positive Ricci curvature. This proof represents one of the famous "vanishing theorems" of the Bochner type, based on a decomposition formula similar to Bochner's formula (Rosenberg, 1997; Gilkey, 1995). By using the Weitzenböck formula, a simple proof of the fundamental elliptic estimate (Garding's inequality) can be achieved (see again, for instance, Rosenberg, 1997, and Gilkey, 1995). Also, the heat kernel proof of the Chern–Gauss–Bonnet theorem, which is known as a special example of the far more general Atiyah–Singer index theorem, makes extensive use of formula (12).

As a second example, we discuss the twisted spin Dirac operator. Let therefore (M, g) denote a spin manifold and $E \xrightarrow{\pi_E} M$ a smooth vector bundle with connection ∇^{E}. Then $\mathscr{E} := \mathrm{S} \otimes \mathrm{E} \xrightarrow{\pi} M$, with S denoting the (total space of the) spinor bundle (once we have fixed the spin structure), naturally becomes a Clifford module bundle. Indeed, as mentioned in the last section, $Cl(M)_{\mathbb{C}} \simeq \mathrm{End(S)}$ and the Clifford action γ is defined by left multiplication. The twisted spin Dirac operator is defined, correspondingly, as $\not{\partial}_A = \gamma(\nabla^{\mathrm{S}\otimes\mathrm{E}})$. The decomposition of $(\gamma(\nabla^{\mathrm{S}\otimes\mathrm{E}}))^2$ is given by the *Lichnerowicz formula* (Lichnerowicz, 1963; Lawson and Michelson, 1997; Berline *et al.*, 1996),

$$(\gamma(\nabla^{\mathrm{S}\otimes\mathrm{E}}))^2 = -\mathrm{tr}_g\, \nabla^{T^*M\otimes\mathscr{E}}\, \nabla^{\mathscr{E}} + \mathscr{R} \tag{14}$$

$$\mathscr{R} := \tfrac{1}{4} r_M + \gamma(\mathrm{F}_{\nabla}^{\mathscr{E}/\mathrm{S}}) \tag{15}$$

Here, $\mathrm{F}_{\nabla}^{\mathscr{E}/\mathrm{S}} := \mathrm{F}_{\nabla}^{\mathscr{E}} - \mathrm{R}^{\mathrm{S}} \otimes \mathrm{Id}_{\mathrm{E}}$ denotes the *relative curvature*, where $\mathrm{F}_{\nabla}^{\mathscr{E}}$ is the total curvature on $\mathscr{E} = \mathrm{S} \otimes \mathrm{E}$ with respect to the connection $\nabla^{\mathrm{S}\otimes\mathrm{E}}$ and R^{S} denotes the curvature on the spinor bundle with respect to the spin connection ∇^{S}. In the case at hand, the relative curvature is fully determined by the curvature F^{E} of the vector bundle E with respect to the connection ∇^{E}. For this reason, the relative curvature is also called the *twisting curvature*. It is this point which becomes different when more general Dirac operators are considered (see below). In what follows, we will denote by $\mathrm{F}_{\nabla}^{\mathscr{E}/\mathrm{S}}$ the relative curvature associated with a connection $\nabla^{\mathscr{E}}$ on the Clifford module $\mathscr{E}$. If the connection is a Clifford connection $\nabla^{\mathscr{E}} \equiv \partial_A \in \mathscr{A}_{Cl}(\mathscr{E})$, we then denote its relative curvature by $\mathrm{F}^{\mathscr{E}/\mathrm{S}}$. In any case, the latter is called the twisting curvature of the Clifford connection ∂_A independent of whether $\mathscr{E}$ denotes a twisted spinor bundle or not. Note that this makes perfect sense because of the fundamental decomposition (5). The decomposition formulas (12) and (14) have in common that they are "Hamiltonian-like." That is, they consist of a second-order operator and a "potential." There is no first-order operator involved! Note that the second-order operator of the previous example $\nabla^*\nabla$ is actually of the same form as in the second example. Indeed, it can be rewritten as $\nabla^*\nabla = -\mathrm{tr}_g\, \nabla^{T^*M\otimes\mathscr{E}}\, \nabla^{\mathscr{E}}$, with $\mathscr{E} := \Lambda^*(T^*M)$. This is a very general feature. In fact, any generalized Laplacian H acting on sections into a smooth vector bundle $\mathscr{E} \xrightarrow{\pi} M$ can be shown to decompose as follows (Gilkey, 1995; Berline *et al.*, 1996; Branson *et al.*, 1998 Esposito, 1998):

$$\mathrm{H} = \Delta^{\nabla^{\mathscr{E}}} + \mathscr{V} \tag{16}$$

The second-order operator is called the *Bochner Laplacian*. It is defined by a connection $\nabla^{\mathscr{E}}$ on the vector bundle $\mathscr{E}$ as $\Delta^{\nabla^{\mathscr{E}}} := -\mathrm{tr}_g\, \nabla^{T^*M\otimes\mathscr{E}}\, \nabla^{\mathscr{E}}$. Both the connection $\nabla^{\mathscr{E}}$ and the zero-order operator $\mathscr{V} \in \Omega^0(M, \mathrm{End}(\mathscr{E}))$ are uniquely defined by the generalized Laplacian H. Note that the formula (16) is also sometimes called the Lichnerowicz decomposition. In order to prove formulas

(12) and (14), one makes extensive use of the fact that, respectively, the Gauss–Bonnet operator and the spin Dirac operator are associated to the Clifford connection $\nabla^{\mathscr{E}} \equiv \nabla^{\Lambda^*(T^*M)}$ on the Clifford module $\mathscr{E} := \Lambda^*(T^*M)$ and $\nabla^{\mathscr{E}} \equiv \nabla^{S\otimes E}$ on the twisted spinor module $\mathscr{E} := S \otimes E$. As a consequence, the Bochner Laplacian is also defined with respect to the corresponding Clifford connection. Indeed, let $(\mathscr{E}, \gamma) \xrightarrow{\pi} (M, g)$ be an arbitrary Clifford module bundle and $\not{\partial}_A$ a twisted Dirac operator. Then we know that there exists a unique Clifford connection ∂_A on $\mathscr{E}$ which is defined by some given $A \in \Omega^1(M, \text{End}^+_{CL}(\mathscr{E}))$ such that the Lichnerowicz decomposition reads (see below)

$$(\not{\partial}_A)^2 = \Delta^{\partial_A} + \mathscr{V} \tag{17}$$

Here the Bochner Laplacian is defined by the Clifford connection, which also defines the Dirac operator, and the potential is as in (15). Also in this more general case, the relative curvature (also denoted by $F^{\mathscr{E}/S}_{\partial_A}$) is fully determined by the "twisting part" of the connection, i.e., by A. However, as a rule, this does not hold true. By use of (16), we know that a unique connection $\hat{\nabla}^{\mathscr{E}}$ on $\mathscr{E}$ and a unique potential $\mathscr{V}$ also exist for an arbitrary Dirac operator D on an arbitrary Clifford module bundle $\mathscr{E}$, such that

$$D^2 = \Delta^{\hat{\nabla}^{\mathscr{E}}} + \mathscr{V} \tag{18}$$

However, neither is the connection $\hat{\nabla}^{\mathscr{E}}$ a representative of the Dirac operator D nor is the relative curvature in this most general case fully determined just by the twisting part of some connection representing the Dirac operator at hand. Therefore, given an arbitrary Dirac operator D, one may ask how to find the connection that defines the corresponding Bochner Laplacian, and how to find an explicit formula that determines the potential $\mathscr{V}$? This question has been answered (e.g., in Ackermann and Tolksdorf, 1996a, b; see also. Bismut, 1989, and Tolksdorf, 1998). Note that in the most general case, the proof of (16) is either local or not very explicit. In particular, there is no explicit formula that determines the potential of the given generalized Laplacian (see, however, Branson *et al.*, 1998). In the remainder of this section we will discuss a formula introduced in (Ackermann and Tolksdorf, 1996a, b). This formula transparently generalizes the Lichnerowicz formula (14) to arbitrary Dirac operators. It allows us to determine both the Bochner Laplacian and the potential of a given Dirac operator on an arbitrary Clifford module bundle. We close this section with a brief discussion of the twisted spin Dirac operator with torsion.

Let again $(\mathscr{E}, \gamma) \xrightarrow{\pi} (M, g)$ denote a Clifford module bundle over a (pseudo) Riemannian manifold. Also, let $\tilde{D} \in \mathscr{D}(\mathscr{E})$ be a Dirac operator compatible with the Clifford action γ [see the definition (4) of the previous section]. Moreover, let $\partial_A \in \mathscr{A}_{Cl}(\mathscr{E})$ be an arbitrary Clifford connection, defined by a one-form $A \in \Omega^1(M, \mathrm{End}^+_{Cl}(\mathscr{E}))$. Then, by use of the mapping (8), we introduce the natural representative $\tilde{\nabla}^{\mathscr{E}} := \partial_A + \delta_\xi(\tilde{D} - \not{\partial}_A)$ of the Dirac operator at hand, such that $\tilde{D} = \not{\nabla}^{\mathscr{E}}$. We obtain, respectively, for the connection defining the Bochner Laplacian and the potential of the generalized BLW decomposition formula (18)

$$\tilde{\nabla}^{\mathscr{E}} := \tilde{\nabla}^{\mathscr{E}} + \omega_{\tilde{\nabla}^{\mathscr{E}}} \tag{19}$$

$$\mathscr{V} := \gamma(\mathrm{F}^{\mathscr{E}}_{\tilde{\nabla}}) + \mathrm{tr}_g(\tilde{\nabla}^{T^*M\otimes\mathrm{End}(\mathscr{E})}\omega_{\tilde{\nabla}^{\mathscr{E}}} + \omega^2_{\tilde{\nabla}^{\mathscr{E}}}) \tag{20}$$

where the one form $\omega_{\tilde{\nabla}^{\mathscr{E}}} \in \Omega^1(M, \mathrm{End}^+(\mathscr{E}))$ is locally given by

$$\omega_{\tilde{\nabla}^{\mathscr{E}}} := -\tfrac{1}{2}g(e_\mu, e_\nu)e^\mu \otimes \gamma(e^\lambda)([\tilde{\nabla}^{\mathscr{E}}{}_\gamma, \gamma(e^\nu)] + \Gamma^\nu{}_{\sigma\lambda}\gamma(e^\sigma)) \tag{21}$$

The Γ's denote the Christoffel symbols defined by the metric g and (e_μ) denotes any given local frame in TM with the dual frame (e^μ) in $T^*M \hookrightarrow Cl(M)$. It can be shown that the endomorphism $\mathscr{V}$ actually only depends on the connection class defining the Dirac operator at hand (Tolksdorf, 1998). The formula (18) obviously generalizes the Lichnerowicz formula (14) to arbitrary Dirac operators. In fact, if $\tilde{\nabla}^{\mathscr{E}} \equiv \partial_A \in \mathscr{A}_{Cl}(\mathscr{E})$ denotes a Clifford connection, it satisfies the fundamental relation (6), and thus the one-form $\omega_{\tilde{\nabla}^{\mathscr{E}}}$ vanishes. Note that also in the general case (18), the potential decomposes as

$$\mathscr{V} = \frac{r_M}{4} + \gamma(\mathrm{F}^{\mathscr{E}/\mathrm{S}}) + \text{“correction” terms involving } \omega$$

Therefore, in the case of a spin Dirac operator $\not{\nabla}^{\mathrm{S}\otimes\mathrm{E}}$ on a twisted spinor bundle, the potential (20) reduces to $\mathscr{R}$ [see formula (15)]. As already stated, for the same reason, this holds true even in the case of a twisted Dirac operator $\not{\partial}_A$ on an arbitrary Clifford module bundle. Consequently, the formula (18) generalizes the Lichnerowicz decomposition (14) to arbitrary Dirac operators. Note again that in the case of a twisted Dirac operator, the corresponding Bochner Laplacian is defined with respect to the same Clifford connection that represents the Dirac operator. Formula (20) is an immediate consequence of the identity

$$\tilde{\mathrm{D}}^2 = \Delta^{\partial_A} - \mathrm{tr}_g(\alpha\partial_A) + \mathscr{V}' \tag{22}$$

with $\mathscr{V}' := \gamma(F^{\mathscr{E}}_{\partial_A}) + \gamma(\nabla^{T^*M\otimes\mathrm{End}(\mathscr{E})}\omega) + \gamma(\omega)^2$. Here, the one-form α is locally defined by $\alpha(e_\mu) := 2\omega(e_\mu) - g(e_\mu, e_\nu)\gamma(e^\sigma)[\omega(e_\sigma), \gamma(e^\nu)]$. The identity (22) shows that the Bochner Laplacian may even be defined by a Clifford connection ∂_A though the potential does not reduce to (15). As it turns out, this is the case for the Dirac–Yukawa operator discussed below. Therefore, this Dirac operator is distinguished in that its BLW decomposition reduces to

$$\tilde{D}^2 = \nabla^{\partial_A} + \mathcal{V}' \tag{23}$$

For a general discussion of the particular case $\alpha = 0$, $\omega \neq 0$ we refer to Ackermann and Tolksdorf, 1996a, b). Note, however, that the result also depends on the tensor product chosen in (5). Next we will discuss an example of a Dirac operator which is not associated with a Clifford connection. The example we consider is the spin Dirac operator with torsion. Let therefore once more (M, g) be an even-dimensional spin manifold with spin connection ∇^S. The corresponding Dirac operator is simply denoted by $\partial\!\!\!/$. In other words, the Clifford module $\mathcal{E}$ we are considering here is just the spinor module once we have chosen an appropriate spin structure. The Clifford relations are respected by any metric connection $\tilde{\nabla}^{T^*M}$ on T^*M. Thus a a metric connection can be lifted to a connection denoted by $\tilde{\nabla}^S$ on the Clifford module $\mathcal{E} \equiv S$. Note that this is in fact the most general connection on S, which is the lift of any given connection on the (co-) tangential bundle. It is well known that the set of metric connections on (M, g) is an affine set which is modeled on the vector space of *torsions* (Spivak, 1979). Here, the affine set may be identified with $\Omega^2(M, T^*M)$. Thus $D := \partial\!\!\!/ + \gamma(\tau)$ is the most general Dirac operator associated with an arbitrary metric connection on T^*M, where, respectively, γ and $\tau \in \Omega^2(M, T^*M)$ denote the (canonical) Clifford action and an arbitrary torsion. Correspondingly, we may define $\omega := \delta_\xi(\gamma(\tau))$ and calculate in this case the connection defining the Bochner Laplacian and the potential $\mathcal{V}$ of D. The corresponding calculations are similar to those presented in Ackermann and Tolksdorf, 1996a, b) and are therefore omitted here. For more general first-order operators, an analogous calculation was carried out by Weitzenböck, (1923). Note that the Dirac operator $D = \partial\!\!\!/ + \gamma(\tau)$ is not represented by a Clifford connection on S since the only Clifford connection in the case at hand is the spin connection defining $\partial\!\!\!/$.

In the following section, we will consider two Dirac operators inspired by physics and not associated with Clifford connections. We will discuss these Dirac operators in some detail since they are important in the context of a geometrical description of the Standard Model of particle physics in terms of Dirac operators (Tolksdorf, 1998).

5. THE PAULI–DIRAC OPERATOR

In this section, we will discuss the Pauli–Dirac operator and compare it with the Dirac–Yukawa operator. We then consider a mixture of both operators and discuss its corresponding BLW decomposition, which has a generic form. The Pauli–Dirac and Dirac–Yukawa operators play a fundamental role in a geometrical description of the Standard Model of particle physics. In particular, the mixture of both allows us geometrically to describe the full action functional of the Standard Model with (Euclidean) gravity included. We will show in this part of the paper that the Pauli–Dirac operator is related to charge in a similar way that the Dirac–Yukawa operator is related to mass. Just as the Dirac–Yukawa operator becomes a "true" Dirac operator when the weak interaction is taken into account, the Pauli–Dirac operator becomes a "true" Dirac operator when charge conjugation is taken into account. This section also intends to give an interpretation of the Dirac operator introduced in Tolksdorf (1998) (which we will call the Pauli–Dirac–Yukawa operator), which permits a geometric description of a certain class of gauge theories within the Clifford frame.

Let $(\mathcal{E}, \gamma) \xrightarrow{\pi} (M, g)$ be a smooth $\mathbb{Z}_2$-graded Hermitian Clifford module bundle over a smooth Riemannian manifold of even dimension. Let us denote, respectively, the grading involution and the Hermitian product on the vector bundle $\mathcal{E}$ by $\chi_{\mathcal{E}}$ and $\langle\cdot, \cdot\rangle_{\mathcal{E}}$. We assume the Clifford action to be Hermitian. The induced "Hermitian" form on $\Gamma(\mathcal{E})$ is denoted by $\langle\cdot, \cdot\rangle_{\Gamma(\mathcal{E})}$. Moreover, let $\partial_A \in \mathcal{A}_{Cl}(\mathcal{E})$ denote any (Hermitian) Clifford connection and let $\mathrm{F}^{\mathcal{E}/\mathrm{S}}$ be the corresponding relative curvature on $\mathcal{E}$. We will consider the operator

$$\mathrm{D} := \not{\partial}_A + i\gamma(\mathrm{F}^{\mathcal{E}/\mathrm{S}}) \tag{24}$$

which we call the *Pauli–Dirac operator.* We assume $\not{\partial}_A$ to be symmetric (which is no restriction). Note that $\gamma(\mathrm{F}^{\mathcal{E}/\mathrm{S}})$ is Hermitian and thus the operator (24) is not symmetric, which will be important. However, on $\Gamma(\mathcal{E})$ this operator is not a Dirac operator, since its zero-order part ("Pauli term") is even with respect to the grading involution $\chi_{\mathcal{E}}$. Note that this is similar to the operator originally introduced by Dirac: $\not{\partial} - m$ (see our historical review). Here, the "mass" term m is to be considered an even operator and thus $\not{\partial} - m$ is not a Dirac operator. The Clifford module in this case is a (trivial) twisted spinor bundle with, say, $\mathrm{E} = \mathbb{C}$. To turn $\not{\partial} - m$ into a Dirac operator, one uses a simple mathematical trick. One just doubles the degrees of freedom to obtain an additional grading. Therefore, one may consider the operator $\not{\partial} - \Phi$ on the Clifford module $\tilde{\mathcal{E}} := \mathcal{S} \otimes (\mathrm{E} \otimes \mathrm{E})$, where $\Phi := \mathrm{Id}_\mathrm{S} \otimes \mathcal{D}$ and $\mathcal{D}$ denotes an off-diagonal matrix with entries equal to m. With these conditions taken into account, the operator $(\not{\partial} - \Phi)$ becomes indeed a Dirac operator on $\tilde{\mathcal{E}}$. If we take "diagonal sections," $\Psi := (\psi, \psi)$ leads to the action $\langle\Psi, (\not{\partial} - \Phi)\Psi\rangle_{\Gamma(\tilde{\mathcal{E}})} = 2\langle\psi, (\not{\partial} - m)\psi\rangle_{\Gamma(\mathcal{E})}$. Of course, this can be generalized to arbitrary twisted spinor bundles with a $\mathbb{Z}_2$-graded internal space, i.e., $\mathrm{E} = \mathrm{E}_\mathrm{L} \oplus \mathrm{E}_\mathrm{R}$. For reasons of covariance, the operator $\not{\partial}$ is replaced by the covariant Dirac operator $\not{\partial}_A$ and the "internal Dirac operator" is replaced by an arbitrary odd endomorphism $\phi \in \Omega^0(M, \mathrm{End}^-(\mathrm{E}))$. Finally, in order not to double count the spin degrees of freedom, one replaces $\mathrm{Id}_\mathrm{S} \otimes \phi$ by $\chi_\mathrm{S} \otimes \phi$. Here, χ_S denotes the grading involution on the spinor bundle. The well-known operator

$$D := \not{\partial} - \Phi \tag{25}$$

is called the *Dirac–Yukawa operator.* The crucial point is that the Dirac–Yukawa operator is Hermitian and thus the corresponding action $\langle \Psi, D\Psi \rangle_{\Gamma(\tilde{\mathscr{E}})}$ is real. Therefore, one cannot get rid of the additional "Yukawa coupling" terms by demanding the action to be real. Of course, this is not a drawback since these additional couplings should give rise to the masses of the fermions. However, what is still left is the physical interpretation of the doubling of the internal degrees of freedom. Fortunately, nature gives us the answer. It is well known that there are two kinds of fermions: left-handed and right-handed fermions. So, we may interpret, respectively, E_L and E_R as the spaces of internal degrees of freedom of the left-handed and the right-handed fermions, respectively. Note that if the neutrinos are massless, both vector spaces are of different dimension. As a result, we find that it is very reasonable to consider the Yukawa coupling Φ as defining a new Dirac operator (25) by doubling the internal degrees of freedom. This is well known.

Now let us return to the operator (24). Since the problem with this operator is the same as in the case of the mass term in the original Dirac operator, we are tempted to try the same trick. So let us double the internal degrees of freedom once more, i.e., let us consider the Clifford module $\tilde{\mathscr{E}} := S \otimes (E \oplus E)$. Note that F denotes the curvature associated with the connection form $A \in \Omega^1(M, \text{End}^+(E))$. Consequently, this time, the doubling of the internal degrees of freedom must be realized using the same vector bundle E. Indeed, the operator (24) becomes a Dirac operator on $\tilde{\mathscr{E}}$ with respect to the grading involution $\chi_{\tilde{\mathscr{E}}}(\psi_1, \psi_2) := (\chi_{\mathscr{E}}(\psi_1), -\chi_{\mathscr{E}}(\psi_1))$ for all $(\psi_1, \psi_2) \in \tilde{\mathscr{E}}$. Let us restrict ourselves to diagonal sections again, $\Psi := (\psi, \psi)$. This leads to the action $\langle \Psi, D\Psi \rangle_{\Gamma(\tilde{\mathscr{E}})} = 2\langle \psi, \not{\partial}_A \psi \rangle_{\Gamma(\mathscr{E})}$. This time, the additional coupling term (the "Pauli term") drops out. And this is what we want to have since such a coupling leads to inconsistent results, as is well known.

As in the case of the Dirac–Yukawa operator, we must find a satisfying interpretation of the doubling of the internal degrees of freedom and why it is reasonable to restrict the fermionic action to diagonal sections. We will show that this is actually obtained by introducing the concept of *charge conjugation*. The crucial point is that the operator (24) is not Hermitian. Indeed, by considering $\tilde{\mathscr{E}} = \mathscr{E} \otimes \mathscr{E} = \mathbb{C} \otimes_{\mathbb{R}} \mathscr{E}$ as the complexification of $\mathscr{E}$, we get $D^* = \overline{D} = \not{\partial}_A - i\gamma(F^{\mathscr{E}/S})$. Of course, on $\mathbb{C} \otimes_{\mathbb{R}} \mathscr{E}$, there exists a canonical *real structure* $\mathscr{T}'$ given by $\mathscr{T}'(\psi_1, \psi_2) := (\psi_1, -\psi_2)$. Alternatively, we may write $\Psi = \psi_1 + i\psi_2$ and thus $\overline{\Psi} \equiv \mathscr{T}'(\Psi) = \psi_1 - i\psi_2$. In order to obtain a real action, we consider the functional

$$2\text{Re}(\langle \Psi, D\Psi \rangle_{\Gamma(\tilde{\mathscr{E}})}) = \langle \Psi, D\Psi \rangle_{\Gamma(\tilde{\mathscr{E}})} + \overline{\langle \Psi, D\Psi \rangle}_{\Gamma(\tilde{\mathscr{E}})}$$

Consequently, the Pauli term drops out. Moreover, when we restrict ourselves to real sections, we obtain

$$\mathrm{Re}(\langle \Psi, \mathrm{D}\Psi \rangle_{\Gamma(\tilde{\mathscr{E}})}) = 2\langle \psi, \not{\partial}_A \psi \rangle_{\Gamma(\mathscr{E})} \tag{26}$$

In contrast to the vector bundle E, the spinor bundle S always admits a real structure, denoted by $\mathscr{T}_{\mathrm{S}}$. We thus obtain an appropriate real structure $\mathscr{T}$ on $\tilde{\mathscr{E}}$ by $\mathscr{T} := \mathscr{T}_{\mathrm{S}} \otimes \mathscr{T}_{\mathrm{E}}$. The action of $\mathscr{T}$ is as follows: $\mathscr{T}(\Psi) = \mathscr{T}_{\mathrm{S}}(\overline{\Psi})$. Here the latter notation means $\Sigma_a \mathscr{T}_{\mathrm{S}}(s^a) \otimes \overline{\Phi}_a$, where (s^a) denotes any given (local) spin frame in the spinor bundle S and $\Phi_a \equiv \phi_{1a} + i\phi_{2a}$ denotes the internal degrees of freedom on $\tilde{\mathrm{E}} := \mathbb{C} \otimes_{\mathbb{R}} \mathrm{E}$ (i.e., $\tilde{\mathscr{E}} = \mathrm{S} \otimes \tilde{\mathrm{E}}$). Finally, to obtain a symmetric form of the Pauli–Dirac operator (24) with respect to charge conjugation, we note that $\tilde{\mathrm{E}} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{E} \simeq \mathrm{E} \otimes \overline{\mathrm{E}}$, where $\overline{\mathrm{E}}$ denotes the complex conjugate bundle of E. We then use the following notation: $\Psi := (\psi, 0)$, $\overline{\Psi} := (0, \overline{\psi}) \in (\mathrm{S} \otimes \mathrm{E}) \otimes (\mathrm{S} \otimes \overline{\mathrm{E}})$. The action of the Pauli–Dirac operator (24) on, respectively, Ψ and $\overline{\Psi}$ is appropriately defined as $\mathrm{D}\Psi := (\mathrm{D}\psi, 0)$ and $\mathrm{D}\overline{\Psi} := 0$. Moreover, we denote by $\Psi^c \equiv \mathscr{T}(\Psi)$ the "charge conjugate" spinor of Ψ. Next we consider the Dirac operator

$$\tilde{\mathrm{D}} := \mathrm{D} + \mathscr{T}\mathrm{D}\mathscr{T}^{-1} \tag{27}$$

This operator is obviously invariant with respect to charge conjugation $\mathscr{T}$. Moreover, when restricted to the real subvector bundle of "particles and antiparticles"

$$\mathscr{E} \otimes \mathscr{E}^c := \{\tilde{\Psi} := \Psi + \Psi^c | \mathscr{T}(\Psi) = \Psi^c\} \subset \mathscr{E} \otimes \mathscr{E} \tag{28}$$

the operator (27) becomes essentially a doubling of the Pauli–Dirac operator (24). Correspondingly, the fermionic action reads

$$\begin{aligned} \langle \tilde{\Psi}, \tilde{\mathrm{D}}\tilde{\Psi} \rangle_{\Gamma(\tilde{\mathscr{E}})} &= \langle \psi, \mathrm{D}\psi \rangle_{\Gamma(\mathscr{E})} + \overline{\langle \psi, \mathrm{D}\psi \rangle}_{\Gamma(\mathscr{E})} \\ &= 2\langle \psi, \not{\partial}_A \psi \rangle_{\Gamma(\mathscr{E})} \end{aligned} \tag{29}$$

We have shown how the concepts of mass and charge are reflected in appropriate $\mathbb{Z}_2$-gradings when seen from the perspective of Dirac operators. Indeed, when charge conjugation is taken into account, one may consider the operator (24) as a Dirac operator.

In what follows, we will give a certain mixture of the Dirac–Yukawa operator and the Pauli–Dirac operator, called the Pauli–Dirac–Yukawa (PDY) operator. As mentioned before, this operator plays a significant role in the geometrical description of the Standard Model of particle physics within the Clifford frame. We also give the BLW decomposition of the "real form" of the PDY operator which will be defined below. For this, we denote by $\tilde{\nabla}^{\mathscr{E}} = \partial_A - \omega_\Psi$ the natural connection associated with the Dirac–Yukawa operator (25), which we denote by D. The corresponding relative curvature of the connection $\tilde{\nabla}^{\mathscr{E}}$ is denoted by $\mathrm{F}_{\tilde{\nabla}}^{\mathscr{E}/\mathrm{S}}$. We consider the Dirac operator

$$\tilde{\mathrm{D}} := \mathrm{D} + i\gamma(\mathrm{F}_{\tilde{\nabla}}^{\mathscr{E}/\mathrm{S}}) \tag{30}$$

which is called the *Pauli–Dirac–Yukawa* operator (PDY). According to our discussion above, this operator is indeed a Dirac operator on $\tilde{\mathscr{E}}$. Alternatively, one may consider the "charge symmetric form" of (30): $\tilde{\mathrm{D}} + \mathscr{T}\tilde{\mathrm{D}}\mathscr{T}^{-1}$ on $\tilde{\mathscr{E}}$.

Instead of considering the BLW decomposition of (30) in terms of our formula (20), we consider the BLW decomposition of the real operator $|\tilde{\mathrm{D}}|^2 = \frac{1}{2}(\tilde{\mathrm{D}}\,\tilde{\mathrm{D}}^* + \tilde{\mathrm{D}}^*\tilde{\mathrm{D}})$, which we call the "real form" of $\tilde{\mathrm{D}}$. We obtain

$$|\tilde{\mathrm{D}}|^2 = \mathrm{D}^2 + (\gamma(\mathrm{F}_{\tilde{\nabla}}^{\mathscr{E}/\mathrm{S}}))^2 \tag{31}$$

Note that this is a positive operator, in contrast to the operator $\tilde{\mathrm{D}}^2$, which was considered in Toiksdorf (1998) to geometrically describe the Standard Model.

In the case of the Pauli–Dirac operator, we obtain the BLW decomposition of the real form of (24)

$$|\tilde{\mathrm{D}}|^2 = \nabla^{\partial_A} + \frac{1}{4}r_M + \gamma(\mathrm{F}^{\mathscr{E}/\mathrm{S}}) + (\gamma(\mathrm{F}^{\mathscr{E}/\mathrm{S}}))^2 \tag{32}$$

where we made use of the Lichnerowicz formula (15).

Finally, in the case of the Pauli–Dirac–Yukawa operator, we obtain the following BLW decomposition of (30):

$$|\tilde{\mathrm{D}}|^2 = \Delta^{\partial_A} + \frac{1}{4}r_M + \gamma(\mathrm{F}^{\mathscr{E}/\mathrm{S}}) + (\gamma(\mathrm{F}_{\tilde{\nabla}}^{\mathscr{E}/\mathrm{S}}))^2 - \chi_{\mathrm{S}} \otimes \nabla^{\mathrm{End(E)}}\phi + \mathrm{Id}_{\mathrm{S}} \otimes \phi^2 \tag{33}$$

Note that the relative curvature $\mathrm{F}_{\tilde{\nabla}}^{\mathscr{E}/\mathrm{S}}$ is defined with respect to the natural connection $\tilde{\nabla}^{\mathscr{E}} := \partial_A + \omega_\Phi$ defining $\tilde{\mathrm{D}}$ and is thus more complicated than the twisting curvature defined by ∂_A! We have made use of the fact that the Bochner Laplacian of the Dirac–Yukawa operator is also defined by the Clifford connection ∂_A. Consequently, the potential $\mathscr{V}$ in the BLW decomposition of D^2 reduces to $\mathscr{V}'$ [see formula (23)]. Moreover, we have used that the canonical one-form ξ is covariantly constant with respect to any given Clifford connection.

We finish this section with several remarks: First, the above constructions can be generalized to arbitrary Clifford module bundles $(\mathscr{E}, \gamma) \xrightarrow{\pi} M^{(p,q)}$. However, for a well-defined charge conjugation operator to exist, we have to assume that (M, g) has at least a spin$^{\mathbb{C}}$-structure. Second, since the imaginary part of $\tilde{\mathrm{D}}^*\tilde{\mathrm{D}}$ (or of $\tilde{\mathrm{D}}\,\tilde{\mathrm{D}}^*$) is a "traceless" term, it is sufficient to consider only the real form of $\tilde{\mathrm{D}}$ insofar as the trace of $\tilde{\mathrm{D}}^*\tilde{\mathrm{D}}$ is concerned. In particular, this holds true for the trace of the propagator of $\tilde{\mathrm{D}}^*\tilde{\mathrm{D}}$ [i.e., the Wodzicki residue of $(\tilde{\mathrm{D}}^*\tilde{\mathrm{D}})^{-2}$, where $\dim M = 4$], which gives rise to the bosonic action of the Standard Model of particle physics. Note that the propagator of $\tilde{\mathrm{D}}^*\tilde{\mathrm{D}}$ gives rise to a positive functional, whereas this may not hold true in general for the propagator of $\tilde{\mathrm{D}}^2$. Finally, we note that from a physical perspective, the Pauli–Dirac–Yukawa operator (30) does not yet have the right physical dimension. Instead of (30), the "physical" PDY operator reads

$$\tilde{\mathrm{D}} := \mathrm{D} + i\left(\frac{l}{g}\,\gamma(\mathrm{F}_{\nabla}^{\mathscr{E}/\mathrm{S}})\right) \tag{34}$$

where, respectively, *l* and *g* denote a length scale and a "coupling constant." It turns out that the length scale is fixed by the mass of the Higgs field (Tolksdorf, 1998). We argue that the coupling constant may correspondingly be identified with the electric charge. This will be investigated in more detail in a forthcoming paper.

6. SUMMARY

In this paper, we gave a brief review of some features of generalized Dirac operators. In particular, we discussed the decomposition of the square of an arbitrary Dirac operator. By use of the generalization of the Lichnerowicz formula, we presented, it becomes evident that the Dirac–Yukawa operator is a distinguished operator within the Clifford frame summarized in Section 3. Though not a twisted Dirac operator, the Bochner Laplacian of the Dirac–Yukawa operator is nevertheless defined by a Clifford connection. As we have mentioned before, the Dirac–Yukawa operator also plays a significant role in Connes' noncommutative geometry [besides the corresponding reference given above, see, in particular, Chamseddine and Connes (1996, 1997); for an excellent summary of some of the main features of, respectively, the Standard Model within Connes' noncommutative geometry and Connes' notion of a "real geometry," we refer to Varilly and Gracia-Bondia (1993) and Varilly, (1997)]. In the second part of this paper, we discussed the Pauli–Dirac operator, which is another well-known operator in physics. It turns out to be related to charge in a similar way that the Dirac–Yukawa operator is related to mass. Thus, from the point of view of generalized Dirac operators, the notions of mass as well as charge are reflected in specific graduations of the internal space of fermions. The Pauli–Dirac operator is unitary equivalent to the Dirac operator introduced in Ackermann and Tolksdorf (1996a, b), which gives rise to the combined Einstein–Hilbert–Yang–Mills functional. The Pauli–Dirac and the Dirac–Yukawa operators have a "generic" structure and may be considered as the real and the imaginary parts of a more general type of Dirac operator, which we call the Pauli–Dirac–Yukawa operator and which includes the full Higgs sector of the Lagrangian of the Standard Model. The BLW decomposition of the real form of the Pauli–Dirac–Yukawa operator is straightforward and has to be distinguished from the corresponding BLW decomposition of the square of the Pauli–Dirac–

Yukawa operator, which does not necessarily give rise to a positive functional. Since the PDY operator naturally incorporates both mass and charge (i.e., both ways the fermions can interact), one may hope to find further relations between the notion of charge and mass within the geometry of generalized Dirac operators as reviewed in this paper.

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